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Wavelet Regularization with Error Estimates on a General Sideways Parabolic Equation

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Abstract—A wavelet regularization method for a general sideways parabolic equation is given. Some sharp stability estimates are also provided.

Keywords—Sideways parabolic equation, Ill-posed problem, Regularization, Meyer wavelet, Error estimate.

1. INTRODUCTION

This paper is concerned with the following sideways parabolic equation in the quarter plane [1]:

$$\begin{aligned} &u_t = a(x)u_{xx} + b(x)u_x + c(x)u, && x > 0, \quad t > 0, \\ &u(1,t) = g(t), && t \geqslant 0, \\ &u(x,0) = 0, && x \geqslant 0. \end{aligned} \tag{1.1}$$

Here a, b, and c are given functions such that for some $\lambda, \Lambda > 0$,

$$\begin{gathered} \lambda \leqslant a(x) \leqslant \Lambda, \quad c(x) \leqslant 0, \quad x \in \mathbb{R}^+, \\ a(\cdot) \in C^2\left(\mathbb{R}^+\right), \quad b(\cdot) \in C^1\left(\mathbb{R}^+\right), \quad c(x) \in C\left(\mathbb{R}^+\right). \end{gathered} \tag{1.2}$$

We want to know $u(x,t)$ for $0 \leqslant x < 1$; this is a severely ill-posed problem [1]. Several authors have dealt with the case of heat equation with constant coefficients [2–4]. Numerical methods have been developed also for more general equations [3,5], but, in most cases, the stability theory and convergence proofs have not been generalized accordingly. This paper remedies this by a new wavelet regularization method.

As we consider the problem in $L^2(\mathbb{R})$ with respect to variable t, we extend $u(x,\cdot)$, $g(\cdot)$, $f(\cdot) := u(0,\cdot)$, and other functions appearing in the paper to be zero for $t < 0$. The notations $\|\cdot\|, (\cdot,\cdot)$

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denote L^2-norm and scalar product, respectively, and $\hat{h}(\xi) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-i\xi t} h(t)\,dt$ is the Fourier transform of $h(t)$. The corresponding direct problem with (1.1) is

$$\begin{aligned} u_t &= a(x)u_{xx} + b(x)u_x + c(x)u, && x > 0, \quad t > 0, \\ u(0,t) &= f(t), && t \geqslant 0, \quad f(\cdot) \in L^2(\mathbb{R}), \\ u(x,0) &= 0, && x \geqslant 0. \end{aligned} \tag{1.3}$$

For the uniqueness of solution, we require $\|u(x,\cdot)\|$ be bounded. The following conclusions can be found in [1].

LEMMA 1.1. *Let $v(x,\xi)$ be the solution of the following boundary value problem:*

$$\begin{aligned} i\xi v(x,\xi) &= a(x)v_{xx} + b(x)v_x + c(x)v, && x > 0, \quad \xi \in \mathbb{R}, \\ v(0,\xi) &= 1, \\ \lim_{x\to\infty} v(x,\xi) &= 0, && \xi \neq 0. \end{aligned} \tag{1.4}$$

For $\xi = 0$, we require $v(x,0)$ be bounded as x tends to ∞. Suppose that problem (1.3) has a solution u, then

$$\hat{u}(x,\xi) = v(x,\xi)\hat{f}(\xi), \qquad x > 0. \tag{1.5}$$

NOTE. From (1.5), we know if problem (1.3) has a solution, then

$$\hat{g}(\xi) = v(1,\xi)\hat{f}(\xi), \tag{1.6}$$

$$\hat{u}(x,\xi) = \frac{v(x,\xi)}{v(1,\xi)}\hat{g}(\xi). \tag{1.7}$$

LEMMA 1.2. *There exist constants c_1, c_2, such that for $x \in [0,1]$ and $|\xi|$ large enough, say $|\xi| \geqslant \xi_0$,*

$$c_1 e^{-A(x)\sqrt{|\xi|/2}} \leqslant |v(x,\xi)| \leqslant c_2 e^{-A(x)\sqrt{|\xi|/2}}, \tag{1.8}$$

where $A(x) = \int_0^x (1/\sqrt{a(s)})\,ds$. Moreover, for $x \in [0,1]$, the right-hand side in (1.8) is valid for all $\xi \in \mathbb{R}$ with another constant c_2.

LEMMA 1.3. *If the boundary value problem*

$$\begin{aligned} a(x)v_{xx} + b(x)v_x + c(x)v = 0, \\ v(0) = 1, \qquad v(x)|_{x\to\infty} \text{ bounded}, \end{aligned} \tag{1.9}$$

has a unique solution, then there exist constants c_1', c_2' such that

$$c_1' e^{-A(1)\sqrt{|\xi|/2}} \leqslant |v(1,\xi)| \leqslant c_2' e^{-A(1)\sqrt{|\xi|/2}}, \qquad \forall \xi \in \mathbb{R}. \tag{1.10}$$

2. REGULARIZATION AND ERROR ESTIMATES

Let $\varphi(t), \psi(t)$ be Meyer scaling and wavelet functions, respectively, then from [6] we know $\operatorname{Supp}\hat{\varphi} = [-(4/3)\pi, (4/3)\pi]$, $\operatorname{Supp}\hat{\psi} = [-(8/3)\pi, -(2/3)\pi] \cup [(2/3)\pi, (8/3)\pi]$, and $\psi_{jk}(t) := 2^{j/2}\psi(2^j t - k)$, $j,\, k \in \mathbb{Z}$ constitute an orthonormal basis of $L^2(\mathbb{R})$ and

$$\operatorname{Supp}\hat{\psi}_{jk}(\xi) = \left[-\frac{8}{3}\pi 2^j, -\frac{2}{3}\pi 2^j\right] \cup \left[\frac{2}{3}\pi 2^j, \frac{8}{3}\pi 2^j\right], \qquad k \in \mathbb{Z}. \tag{2.1}$$

The multiresolution analysis (MRA) $\{V_j\}_{j\in\mathbf{Z}}$ of Meyer wavelet is generated by

$$\begin{aligned} V_j = \overline{\{\varphi_{jk} : k \in \mathbb{Z}\}}, \qquad \varphi_{jk} := 2^{j/2}\varphi\left(2^{j/2}t - k\right), \qquad j,\, k \in \mathbb{Z}, \\ \operatorname{Supp}\hat{\varphi}_{jk}(\xi) = \left[-\frac{4}{3}\pi 2^j, \frac{4}{3}\pi 2^j\right], \qquad k \in \mathbb{Z}. \end{aligned} \tag{2.2}$$

The orthogonal projection of a function $g \in L^2(\mathbb{R})$ on space V_j is given by $P_J g := \sum_{k\in\mathbb{R}} (g, \varphi_{Jk})\varphi_{Jk}$, while $Q_J g := \sum_{k\in\mathbb{Z}} (g, \psi_{Jk})\psi_{Jk}$ denotes the projection on wavelet space W_J with $V_{J+1} = V_J \oplus W_J$. It is easy to see from (2.2) and (2.1) that

$$\widehat{P_J g}(\xi) = 0, \qquad \text{for } |\xi| \geqslant \frac{4}{3}\pi 2^J, \tag{2.3}$$

$$\widehat{Q_j g}(\xi) = 0, \qquad \text{for } j > J \text{ and } |\xi| < \frac{4}{3}\pi 2^J. \tag{2.4}$$

Since $(I - P_J)g = \sum_{j\geqslant J} Q_j g$ and from (2.4), we know

$$((I - P_J)g)\hat{}(\xi) = \widehat{Q_J g}(\xi), \qquad \text{for } |\xi| < \frac{4}{3}\pi 2^J. \tag{2.5}$$

LEMMA 2.1. *(See [7].) Let $\{V_j\}_{j\in\mathbb{Z}}$ be Meyer's MRA and suppose $J \in \mathbb{N}, r \in \mathbb{R}$. Then for all $g \in V_J$, we have*

$$\left\|D^k g\right\|_{H^r} \leqslant C 2^{(J-1)k} \|g\|_{H^r}, \qquad k \in \mathbb{N}, \tag{2.6}$$

where C is a positive constant and $D^k = \frac{d^k g}{dt^k}$.

Let T_x be the operator: $g(t) \longmapsto u(x,t)$ for $0 \leqslant x < 1$ defined by (1.7), i.e.,

$$\widehat{T_x g}(\xi) = \hat{u}(x,\xi) = \frac{v(x,\xi)}{v(1,\xi)}\hat{g}(\xi), \qquad 0 \leqslant x < 1, \tag{2.7}$$

where $u(x,t)$ is the solution of problem (1.1). We can prove the following.

LEMMA 2.2. *Suppose problem (1.9) has a unique solution, $\{V_j\}_{j\in\mathbb{Z}}$ are Meyer's MRA, $J \in \mathbb{N}$, $0 \leqslant x < 1$, $r \in \mathbb{R}$. Then for all $g(t) \in V_J$, we have*

$$\|T_x g\|_{H^r} \leqslant C \exp\left\{2^{(J-1)/2}(A(1) - A(x))\right\} \|g\|_{H^r}. \tag{2.8}$$

PROOF. For convenience, we will denote different constants appearing in the proof by same C. Note that

$$e^{(A(1)-A(x))\sqrt{|\xi|/2}} \leqslant 2\sqrt{2}\left|\cosh\left(\sqrt{i\xi}\,(A(1) - A(x))\right)\right|, \tag{2.9}$$

and from (2.7), (1.8), (1.10), (2.9), and Hölder inequality, we know that for $g \in V_J, J \in \mathbb{N}, r \in \mathbb{R}$ holds

$$\begin{aligned}
\|T_x g\|_{H^r} &= \left(\int_{-\infty}^{\infty} \left|\widehat{T_x g}\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} = \left(\int_{-\infty}^{\infty} \left|\frac{v(x,\xi)}{v(1,\xi)}\hat{g}(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\
&\leqslant C\left(\int_{-\infty}^{\infty} \left|e^{(A(1)-A(x))\sqrt{|\xi|/2}}\hat{g}(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\
&\leqslant C\left(\int_{-\infty}^{\infty} \left|\cosh\left(\sqrt{i\xi}\,(A(1) - A(x))\right)\hat{g}(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\
&= C\left(\int_{-\infty}^{\infty} \left|\sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!}(i\xi)^k \hat{g}(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\
&\leqslant C\sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!}\left(\int_{-\infty}^{\infty} \left|(i\xi)^k \hat{g}(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\
&\leqslant C\sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!}\|D_k g\|_{H^r} \leqslant C\sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!} 2^{(J-1)k}\|g\|_{H^r} \\
&= C\cosh\left(2^{(J-1)/2}(A(1) - A(x))\right)\|g\|_{H^r} \leqslant C\exp\left\{2^{(J-1)/2}(A(1) - A(x))\right\}\|g\|_{H^r}.
\end{aligned}$$

Let $g(\cdot), g_m(\cdot)$ be exact and measured data, respectively, which satisfy

$$\|g - g_m\|_{H^r} \leqslant \varepsilon, \qquad \text{for some } r \leqslant 0. \tag{2.10}$$

Since g_m belongs, in general, to $L^2(\mathbb{R}) \subset H^r(\mathbb{R})$ for $r \leqslant 0$, so r should not be positive. We also need an additional condition $f(t) := u(0,t) \in H^s(\mathbb{R})$ for some $s \geqslant r$, and

$$\|f\|_{H^s} \leqslant M. \tag{2.11}$$

Letting $T_{x,J} := T_x P_J$, we can show it approximates T_x in a stable way for an appropriate choice of $J \in \mathbb{N}$ depending on ε and M.

THEOREM 2.1. *Suppose problem (1.9) has a unique solution, then for every fixed $J \in \mathbb{N}$, problem (1.1) with data g in V_J is well-posed. Suppose (2.10),(2.11) hold, then the problem of calculating $T_{x,J} g_m$ is stable. Furthermore, with*

$$J^* := \left[\log_2\left(2\left(\frac{1}{A(1)}\ln\left(\frac{M}{\varepsilon}\left(\ln\frac{M}{\varepsilon}\right)^{-2(s-r)}\right)\right)^2\right)\right], \tag{2.12}$$

where $[a]$ denotes the largest integer less than or equal to $a \in \mathbb{R}$, then

$$\begin{aligned} &\|T_x g - T_{x,J^*} g_m\|_{H^r} \\ &\leqslant \left(C + (c_2 + c_2' C)\, A(1)^{2(s-r)}\left(\frac{\ln(M/\varepsilon)}{\ln(M/\varepsilon) + \ln(\ln(M/\varepsilon))^{-2(s-r)}}\right)^{2(s-r)}\right) \\ &\qquad \cdot M^{1-(A(x)/A(1))}\,\varepsilon^{(A(x)/A(1))}\left(\ln\frac{M}{\varepsilon}\right)^{-2(s-r)(1-(A(x)/A(1)))}, \end{aligned} \tag{2.13}$$

where C, c_2, c_2' are the constants appearing in (2.8), (1.8), (2.11), respectively.

PROOF. $\|T_x g - T_{x,J} g_m\|_{H^r} \leqslant \|T_x g - T_{x,J} g\|_{H^r} + \|T_{x,J}(g - g_m)\|_{H^r}$.

$$\begin{aligned} \|T_{x,J}(g - g_m)\|_{H^r} &= \|T_x P_J (g - g_m)\|_{H^r} \\ &\leqslant C \exp\left\{2^{(J-1)/2}(A(1) - A(x))\right\} \|P_J(g - g_m)\|_{H^r} \\ &\leqslant C \exp\left\{2^{(J-1)/2}(A(1) - A(x))\right\} \varepsilon. \end{aligned}$$

Note that from (2.7),(2.3), we know

$$\begin{aligned} \|T_x g - T_{x,J} g\|_{H^r} &= \|T_x (I - P_J) g\|_{H^r} \\ &= \left(\int_{-\infty}^{\infty} \left|\frac{v(x,\xi)}{v(1,\xi)}((I - P_J)g)\hat{\ }(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\ &\leqslant \left(\int_{|\xi| \geqslant (4/3)\pi 2^J} \left|\frac{v(x,\xi)}{v(1,\xi)}\hat{g}(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\ &\quad + \left(\int_{|\xi| < (4/3)\pi 2^J} \left|\frac{v(x,\xi)}{v(1,\xi)}((I - P_J)g)\hat{\ }(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \\ &:= \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

From (1.6),(1.8), we know

$$\mathrm{I}_1 \leqslant \left(\int_{|\xi| \geqslant (4/3)\pi 2^J} \left|v(x,\xi)\hat{f}(\xi)\right|^2 \left(1 + |\xi|^2\right)^r d\xi\right)^{1/2} \leqslant c_2 2^{-J(s-r)} \exp\left\{-A(x) 2^{J/2}\right\} \|f\|_{H^s}.$$

From (2.5), (2.7), (2.8), and noting that $Q_J g \in W_J \subset V_{J+1}$, we have

$$\begin{aligned}\mathrm{I}_2 &= \left(\int_{|\xi|<(4/3)\pi 2^J} \left|\frac{v(x,\xi)}{v(1,\xi)}\widehat{Q_J g}(\xi)\right|^2 \left(1+|\xi|^2\right)^r d\xi\right)^{1/2} \leqslant \|T_x Q_J g\|_{H^r} \\ &\leqslant C\exp\left\{2^{J/2}(A(1)-A(x))\right\}\|Q_J g\|_{H^r}.\end{aligned}$$

Note that from (1.6) and (1.10), it is easy to show

$$\|Q_J g\|_{H^r} \leqslant c_2' 2^{-J(s-r)} \exp\left\{-A(1)2^{J/2}\right\}\|f\|_{H^s},$$

and so

$$\mathrm{I}_2 \leqslant c_2' C 2^{-J(s-r)} \exp\left\{-A(x)2^{J/2}\right\}\|f\|_{H^s}.$$

Hence,

$$\begin{aligned}\|T_x g - T_{x,J} g_m\|_{H^r} \leqslant{}& C\exp\left\{2^{(J-1)/2}(A(1)-A(x))\right\}\varepsilon \\ &+ \left(c_2 + c_2' C\right) 2^{-J(s-r)} \exp\left\{-A(x)2^{J/2}\right\}\|f\|_{H^s}.\end{aligned} \tag{2.14}$$

Note the representation of J^* and by a simple computation using (2.14), inequality (2.13) can be obtained. The proof is completed.

REMARK 2.1. Taking $s = r = 0$, we can obtain an L^2-estimate

$$\begin{aligned}\|T_x g - T_{x,J^*} g_m\|_{L^2} &\leqslant \left(1 + c_2 + c_2'\right) C M^{1-(A(x)/A(1))}\varepsilon^{(A(x)/A(1))} \\ &:= C_1 M^{1-(A(x)/A(1))}\varepsilon^{(A(x)/A(1))}.\end{aligned} \tag{2.15}$$

Especially, when $a(x) \equiv 1$, then $A(1) = 1, A(x) = x$. So

$$\|T_x g - T_{x,J^*} g_m\|_{L^2} \leqslant C_1 M^{1-x}\varepsilon^x. \tag{2.16}$$

This result has been obtained by several authors by using different regularization methods for the standard sideways heat equation [2–4] and this estimate is at least close to optimal or it is "order optimal" [4] for the sideways heat equation.

REMARK 2.2. From (2.15),(2.16), we know when $x \to 0^+$, the accuracy of regularized solutions become all progressively lower. At $x = 0$, they merely imply that the errors are bounded by $C_1 M$. But if we take $s - r > 0$, then from (2.13), we know the speed of convergence of the regularized solution is faster, and at $x = 0$,

$$\begin{aligned}\|T_0 g - T_{0,J^*} g_m\|_{H^r} &= \|f - T_{0,J^*} g_m\|_{H^r} \\ &\leqslant \left(C + \left(c_2 + c_2' C\right) A(1)^{2(s-r)} \left(\frac{\ln(M/\varepsilon)}{\ln(M/\varepsilon) + \ln(\ln(M/\varepsilon))^{-2(s-r)}}\right)^{2(s-r)}\right) \\ &\quad \cdot M\left(\ln\frac{M}{\varepsilon}\right)^{-2(s-r)} \to 0, \qquad \text{as } \varepsilon \to 0.\end{aligned}$$

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